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# ELZAKI TRANSFORM OF BESSEL'S FUNCTIONS

**Sudhanshu Aggarwal**
Assistant Professor, Department of Mathematics, National P.G. College Barhalganj, Gorakhpur-273402, U.P., India

## ABSTRACT

In the modern time, Bessel's functions appear in solving many problems of sciences and engineering together with many equations such as heat equation, wave equation, Laplace equation, Schrodinger equation, Helmholtz equation in cylindrical or spherical coordinates. In this paper, we determine Elzaki transform of Bessel's functions. Some applications of Elzaki transform of Bessel's functions for evaluating the integral, which contain Bessel's functions, are given.

***Keywords:*** *Elzaki transform, Convolution theorem, Inverse Elzaki transform, Bessel function.*

## I. INTRODUCTION

Bessel's functions have many applications [3] to solve the problems of mathematical physics, acoustics, radio physics, atomic physics, nuclear physics, engineering and sciences such as flux distribution in a nuclear reactor, heat transfer, fluid mechanics, vibrations, hydrodynamics, stress analysis etc.

Bessel's function of order $n$, where $n\epsilon N$ is given by [1-5,10]

$$J_n(t) = \frac{t^n}{2^n n!}[1 - \frac{t^2}{2.(2n+2)} + \frac{t^4}{2.4.(2n+2)(2n+4)} - \frac{t^6}{2.4.6.(2n+2)(2n+4)(2n+6)} + .....] .....(1)$$

In particular, when $n = 0$, we have Bessel's function of zero order and it is denoted by $J_0(t)$and it is given by the infinite power series

$$J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2.4^2} - \frac{t^6}{2^2.4^2.6^2} + .....(2)$$

For $n = 1$, we have Bessel's function of order one and it is denoted by $J_1(t)$and it is given by

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^2.4} + \frac{t^5}{2^2.4^2.6} - \frac{t^7}{2^2.4^2.6^2.8} + ..(3)$$

Equation (3) can be written as

$$J_1(t) = \frac{t}{2} - \frac{t^3}{2^3.2!} + \frac{t^5}{2^5.2!.3!} - \frac{t^7}{2^7.3!.4!} + ....(4)$$

For $n = 2$, we have Bessel's function of order two and it is denoted by $J_2(t)$ and it is given by

$$J_2(t) = \frac{t^2}{2.4} - \frac{t^4}{2^2.4.6} + \frac{t^6}{2^2.4^2.6.8} - \frac{t^8}{2^2.4^2.6^2.8.10} + .....(5)$$

The Elzaki transform of the function $F(t)$ is defined as [6]:

$$E\{F(t)\} = v\int_0^\infty F(t)e^{-t/v}dt$$
$$= T(v), t \geq 0, 0 < k_1 \leq v \leq k_2 ......(6)$$

where $E$ is Elzaki transform operator.

The Elzaki transform of the function $F(t)$ for $t \geq 0$ exist if $F(t)$ is piecewise continuous and of exponential order. These conditions are only sufficient conditions for the existence of Elzaki transform of the function $F(t)$.

Elzaki et al. [7] defined fundamental properties of Elzaki transform together with applications. HwaJoon Kim [8] gave the time shifting theorem and convolution for Elzaki transform. Elzaki and Ezaki [9] discussed the connections between Laplace & Elzaki transforms. Elzaki and Ezaki [12] used Elzaki transform for solving ordinary differential equation with variable coefficients. The solution of partial differential equations using Elzaki transform was given by Elzaki and Ezaki [13]. Shendkar and Jadhav [14] used Elzaki transform for the solution of differential equations.

The object of the present study is to determine Elzaki transform of Bessel's functions and explain the advantage of Elzaki transform of Bessel's functions for evaluating the integral which contain Bessel's functions.

## II. LINEARITY PROPERTY OF ELZAKI TRANSFORM:

If $E\{F(t)\} = H(v)$ and $E\{G(t)\} = I(v)$ then

$$E\{aF(t) + bG(t)\} = aE\{F(t)\} + bE\{G(t)\}$$

$\Rightarrow E\{aF(t) + bG(t)\} = aH(v) + bI(v),$

where $a$,$b$ are arbitrary constants.

## III. ELZAKI TRANSFORM OF SOME ELEMENTARY FUNCTIONS [6, 7]

| S.N. | $F(t)$ | $E\{F(t)\} = T(v)$ |
|---|---|---|
| 1. | $1$ | $v^2$ |
| 2. | $t$ | $v^3$ |
| 3. | $t^2$ | $2!v^4$ |
| 4. | $t^n, n \in N$ | $n!v^{n+2}$ |
| 5. | $t^n, n >-1$ | $\Gamma(n+1)v^{n+2}$ |
| 6. | $e^{at}$ | $\frac{v^2}{1-av}$ |
| 7. | $sinat$ | $\frac{av^3}{1+a^2v^2}$ |
| 8. | $cosat$ | $\frac{v^2}{1+a^2v^2}$ |
| 9. | $sinhat$ | $\frac{av^3}{1-a^2v^2}$ |
| 10. | $coshat$ | $\frac{v^2}{1-a^2v^2}$ |

## IV. CHANGE OF SCALE PROPERTY OF ELZAKI TRANSFORM:

If $E\{F(t)\} = T(v)$ then

$E\{F(at)\} = \text{v}\int_0^\infty F(at)e^{-t/v}dt$ .........(7)

Put $at = p \Rightarrow adt = dp$ in equation (7), we have

$E\{F(at)\} = \frac{\text{v}}{a}\int_0^\infty F(p)e^{\frac{-p}{av}}dp = \frac{1}{a^2}T(av)$

Thus, if $E\{F(t)\} = T(v)$ then

$E\{F(at)\} = \frac{1}{a^2}T(av)$................(8)

## V. ELZAKI TRANSFORM OF THE DERIVATIVES OF THE FUNCTION $F(t)$ [8, 14]:

If $E\{F(t)\} = T(v)$ then

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a) $E\{F'(t)\} = \frac{T(v)}{v} - vF(0)$
b) $E\{F''(t)\} = \frac{T(v)}{v^2} - vF'(0) - F(0)$

## VI. CONVOLUTION OF TWO FUNCTIONS [11]

Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t) * G(t)$ and it is defined by

$$F(t) * G(t) = F * G = \int_0^t F(x)G(t-x)dx$$
$$= \int_0^t F(t-x)G(x)dx = G * F$$

## VII. CONVOLUTION THEOREM FOR ELZAKI TRANSFORM [7, 8]

If $E\{F(t)\} = H(v)$ and $E\{G(t)\} = I(v)$ then

$$E\{F(t) * G(t)\} = \frac{1}{v}E\{F(t)\}E\{G(t)\} = \frac{1}{v}H(v)I(v)$$

## VIII. INVERSE ELZAKI TRANSFORM

If $E\{F(t)\} = T(v)$ then $F(t)$ is called the inverse Elzaki transform of $T(v)$ and mathematically it is defined as
$F(t) = E^{-1}\{T(v)\}$
where $E^{-1}$ is the inverse Elzaki transform operator.

## IX. INVERSE ELZAKI TRANSFORM OF SOME ELEMENTARY FUNCTIONS

| S.N. | $T(v)$ | $F(t) = E^{-1}\{T(v)\}$ |
|---|---|---|
| 1. | $v^2$ | $1$ |
| 2. | $v^3$ | $t$ |
| 3. | $v^4$ | $\frac{t^2}{2!}$ |
| 4. | $v^{n+2}, n \in N$ | $\frac{t^n}{n!}$ |
| 5. | $v^{n+2}, n > -1$ | $\frac{t^n}{\Gamma(n+1)}$ |
| 6. | $\frac{v^2}{1-av}$ | $e^{at}$ |
| 7. | $\frac{v^3}{1+a^2v^2}$ | $\frac{sinat}{a}$ |
| 8. | $\frac{v^2}{1+a^2v^2}$ | $cosat$ |
| 9. | $\frac{v^3}{1-a^2v^2}$ | $\frac{sinhat}{a}$ |
| 10. | $\frac{v^2}{1-a^2v^2}$ | $coshat$ |

## X. RELATION BETWEEN $J_0(t)$AND $J_1(t)$ [4, 10]

$$\frac{d}{dt}J_0(t) =- J_1(t) \ldots\ldots\ldots.(9)$$

## XI. RELATION BETWEEN $J_0(t)$AND $J_2(t)$ [10]

$$J_2(t) = J_0(t) + 2J_0''(t) \ldots\ldots\ldots(10)$$

## XII. ELZAKI TRANSFORM OF BESSEL'S FUNCTIONS

### a) ELZAKI TRANSFORM OF $J_0(t)$:

Taking Elzaki transform of equation (2), both sides, we have

$$E\{J_0(t)\} = E\{1\} - \frac{1}{2^2}E\{t^2\} + \frac{1}{2^2.4^2}E\{t^4\} - \frac{1}{2^2.4^2.6^2}E\{t^6\} + \ldots..$$

$$= v^2 - \frac{1}{2^2}(2!v^4) + \frac{1}{2^2.4^2}(4!v^6) - \frac{1}{2^2.4^2.6^2}(6!v^8) + \ldots\ldots\ldots$$

$$= v^2\left[1 - \frac{1}{2}(v^2) + \frac{1.3}{2.4}(v^2)^2 - \frac{1.3.5}{2.4.6}(v^2)^3 + \ldots\ldots\ldots\ldots\right]$$

$$= v^2(1+v^2)^{-1/2} = \frac{v^2}{\sqrt{(1+v^2)}} \ldots\ldots..(11)$$

### b) ELZAKI TRANSFORM OF $J_1(t)$:

Taking Elzaki transform of equation (9), both sides, we have

$$E\{J_1(t)\} =- E\{J_0'(t)\} \ldots\ldots\ldots\ldots\ldots..(12)$$

Now applying the property, Elzaki transform of derivative of the function on equation (12), we have

$$E\{J_1(t)\} =- \left[\frac{1}{v}E\{J_0(t)\} - vJ_0(0)\right] \ldots.(13)$$

Using equation (2) and equation (11) in equation (13), we have

$$E\{J_1(t)\} =- \left[\frac{1}{v}.\frac{v^2}{\sqrt{(1+v^2)}} - v\right]$$

$$E\{J_1(t)\} = v - \frac{v}{\sqrt{(1+v^2)}} \ldots\ldots\ldots.(14)$$

### c) ELZAKI TRANSFORM OF $J_2(t)$:

Taking Elzaki transform of equation (10), both sides, we have

$$E\{J_2(t)\} = E\{J_0(t)\} + 2E\{J_0''(t)\} \ldots..(15)$$

Now applying the property, Elzaki transform of derivative of the function and using equation (11) in equation (15), we have

$$E\{J_2(t)\} = \frac{v^2}{\sqrt{(1+v^2)}} + 2\left[\frac{1}{v^2}E\{J_0(t)\} - J_0(0) - vJ_0'(0)\right] \ldots\ldots.(16)$$

Using equation (2), equation (9) and equation (11) in equation (13), we have

$$E\{J_2(t)\} = \frac{v^2}{\sqrt{(1+v^2)}} + 2\left[\frac{1}{v^2}.\frac{v^2}{\sqrt{(1+v^2)}} - 1 + vJ_1(0)\right] \ldots\ldots\ldots\ldots\ldots\ldots(17)$$

Using equation (3) in equation (17), we have

$$E\{J_2(t)\} = \frac{v^2}{\sqrt{(1+v^2)}} + \frac{2}{\sqrt{(1+v^2)}} - 2$$

$$= \frac{v^2 + 2 - 2\sqrt{(1+v^2)}}{\sqrt{(1+v^2)}} \ldots\ldots..(18)$$

**d) ELZAKI TRANSFORM OF $J_0(at)$:**

From equation (11), Elzaki transform of $J_0(t)$ is given by

$$E\{J_0(t)\} = \frac{v^2}{\sqrt{(1+v^2)}}$$

Now applying change of scale property of Elzaki transform, we have

$$E\{J_0(at)\} = \frac{1}{a^2}\left[\frac{(av)^2}{\sqrt{(1+(av)^2)}}\right]$$

$$= \left[\frac{v^2}{\sqrt{(1+a^2v^2)}}\right] \text{.............}(19)$$

**e) ELZAKI TRANSFORM OF $J_1(at)$:**

From equation (14), Elzaki transform of $J_1(t)$ is given by

$$E\{J_1(t)\} = v - \frac{v}{\sqrt{(1+v^2)}}$$

Now applying change of scale property of Elzaki transform, we have

$$E\{J_1(at)\} = \frac{1}{a^2}\left[av - \frac{av}{\sqrt{(1+(av)^2)}}\right]$$

$$= \frac{v}{a}\left[1 - \frac{1}{\sqrt{(1+a^2v^2)}}\right] \text{...............}(20)$$

**f) ELZAKI TRANSFORM OF $J_2(at)$:**

From equation (18), Elzaki transform of $J_2(t)$ is given by

$$E\{J_2(t)\} = \frac{v^2+2-2\sqrt{(1+v^2)}}{\sqrt{(1+v^2)}}$$

Now applying change of scale property of Elzaki transform, we have

$$E\{J_2(at)\} = \frac{1}{a^2}\left[\frac{(av)^2+2-2\sqrt{(1+(av)^2)}}{\sqrt{(1+(av)^2)}}\right]$$

$$= \frac{1}{a^2}\left[\frac{a^2v^2+2-2\sqrt{(1+a^2v^2)}}{\sqrt{(1+a^2v^2)}}\right] \text{.............}(21)$$

## XIII. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Elzaki transform of Bessel's functions for evaluating the integral which contain Bessel's functions.

**APPLICATION:1** Evaluate the integral

$$I(t) = \int_0^t J_0(u)J_0(t-u)du \text{..........}(22)$$

Applying the Elzaki transform to both sides of (22), we have

$$E\{I(t)\} = E\left\{\int_0^t J_0(u)J_0(t-u)du\right\} \text{....}(23)$$

Using convolution theorem of Elzaki transform on (23), we have

$$E\{I(t)\} = \frac{1}{v}E\{J_0(t)\}E\{J_0(t)\}$$

$$= \frac{1}{v}.\frac{v^2}{\sqrt{(1+v^2)}}.\frac{v^2}{\sqrt{(1+v^2)}} = \frac{v^3}{1+v^2} \text{...}(24)$$

Operating inverse Elzaki transform on both sides of (24), we have

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$$I(t)=E^{-1}\left\{\frac{v^3}{1+v^2}\right\}=sint……(25)$$

which is the required exact solution of (22).

**APPLICATION:2** Evaluate the integral

$$I(t)=\int_0^t J_0(u)J_1(t-u)du……….(26)$$

Applying the Elzaki transform to both sides of (26), we have

$$E\{I(t)\}=E\left\{\int_0^t J_0(u)J_1(t-u)du\right\}….(27)$$

Using convolution theorem of Elzaki transform on (27), we have

$$E\{I(t)\}=\frac{1}{v}E\{J_0(t)\}E\{J_1(t)\}$$

$$=\frac{1}{v}.\frac{v^2}{\sqrt{(1+v^2)}}.\left[v-\frac{v}{\sqrt{(1+v^2)}}\right]$$

$$=\frac{v^2}{\sqrt{(1+v^2)}}-\frac{v^2}{1+v^2}……(28)$$

Operating inverse Elzaki transform on both sides of (28), we have

$$I(t)=E^{-1}\left\{\frac{v^2}{\sqrt{(1+v^2)}}\right\}-E^{-1}\left\{\frac{v^2}{1+v^2}\right\}$$

$$=J_0(t)-cost…………….(29)$$

which is the required exact solution of (26).

**APPLICATION:3** Evaluate the integral

$$I(t)=\int_0^t J_1(t-u)du……….(30)$$

Applying the Elzaki transform to both sides of (30), we have

$$E\{I(t)\}=E\left\{\int_0^t J_1(t-u)du\right\}….(31)$$

Using convolution theorem of Elzaki transform on (31), we have

$$E\{I(t)\}=\frac{1}{v}E\{1\}E\{J_1(t)\}$$

$$=\frac{1}{v}.v^2.\left[v-\frac{v}{\sqrt{(1+v^2)}}\right]$$

$$=v^2-\frac{v^2}{\sqrt{(1+v^2)}}……(32)$$

Operating inverse Elzaki transform on both sides of (32), we have

$$I(t)=E^{-1}\{v^2\}-E^{-1}\left\{\frac{v^2}{\sqrt{(1+v^2)}}\right\}$$

$$=1-J_0(t)…………….(33)$$

which is the required exact solution of (31).

## XIV. CONCLUSION

In this paper, we have successfully discussed the Elzaki transform of Bessel's functions. The given applications show that the advantage of Elzaki transform of Bessel's functions to evaluate the integral which contain Bessel's functions.

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